4.1 - 4.2 Exponential and Logarithmic Functions

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Definition (Exponential Function)

An exponential function with base a is a function of the form

$$f(x) = a^x,$$

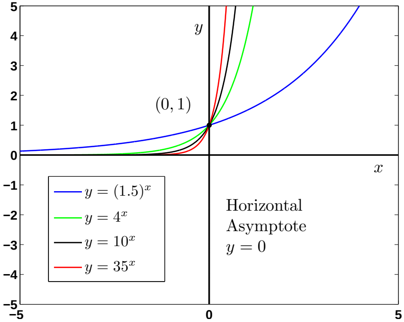
where a and x are real numbers and

- x is the independent VARIABLE of the function; and
- a is a number FIXED CONSTANT such that $a > 0$ and $a \neq 1$.

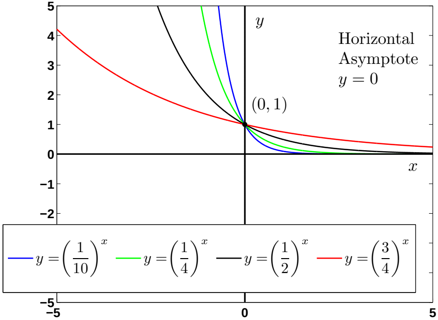
Examples:

$$f(x) = 2^x \qquad g(x) = 10^x$$

$$h(x) = \left(\frac{1}{2}\right)^x \qquad w(x) = \left(\frac{113}{10}\right)^x$$



$a > 1$



$0 < a < 1$

The exponential function $f(x) = a^x$ is only defined for $a > 1$ and $0 < a < 1$, and the graph of an exponential function can only exhibit two types of behavior:

1. exponential growth (if $a > 1$), or
2. exponential decay (if $0 < a < 1$).

Compound Interest Formula

If a principal P (dollars) is invested for t years at an annual rate r, and it is compounded n times per year, then the amount A, or ending balance, is given by

$$A = P\left(1 + \frac{r}{n}\right)^{n \cdot t}$$

Example: If \$23,200 dollars is invested at an interest rate of 7%, find the value of the investment at the end of 5 years if the interest is compounded monthly.

Soln: $P = 23,200$, $r = 7\% = 0.07$, $n = 12$, and $t = 5$. The expected amount in the account after 5 years is

$$A = P\left(1 + \frac{r}{n}\right)^{n \cdot t} = 23,200\left(1 + \frac{0.07}{12}\right)^{12 \cdot 5} = \$32,888.91$$

note: we use a calculator to find the answer here!

Suppose now that \$1 is invested at 100% interest for 1 year (no bank would pay this). The preceding formula becomes a function A defined in terms of the number of compounding periods n:

$$A(n) = \left(1 + \frac{1}{n}\right)^n$$

n	$A(n) = \left(1 + \frac{1}{n}\right)^n$
1 (compounded annually)	\$2.00
2 (compounded semiannually)	\$2.25
3	\$2.370370
4 (compounded quarterly)	\$2.441406
5	\$2.488320
100 (compounded annually)	\$2.704814
365 (compounded daily)	\$2.714567
8760 (compounded hourly)	\$2.718127

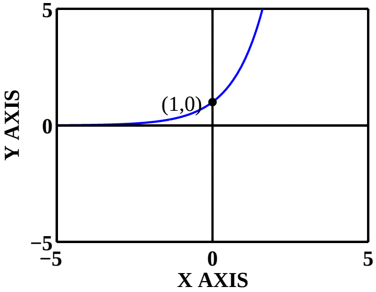
As n gets larger $A(n)$ better approximates the number e.

Definition (The irrational number “e”)

$e \approx 2.7182818284...$

Remember that “e” is a number. We can use e to construct the exponential growth function $f(x) = e^x$.

x	$f(x)$
-2.0	0.1353
-1.0	0.3679
0.0	1.0
1.0	2.7178
2.0	7.3891

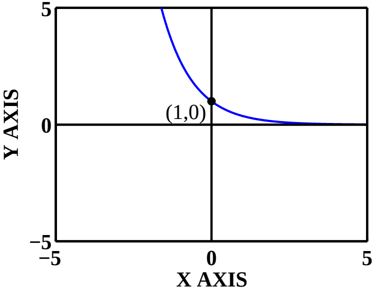


Definition (The irrational number "e")

$e \approx 2.7182818284...$

Moreover e has the multiplicative inverse $e^{-1} = 1/e \approx 0.36787944117...$ We can use e^{-1} to construct the exponential decay function $f(x) = e^{-x}$.

x	$f(x)$
-2.0	7.3891
-1.0	2.7178
0.0	1.0
1.0	0.3679
2.0	0.1353



Continuous Compounding Formula

If a principal P is invested for t years at an annual rate r compounded continuously, then the amount A, or ending balance, is given by

$$A = P \cdot e^{r \cdot t}$$

Example: If \$23,200 dollars is invested at an interest rate of 7%, find the value of the investment at the end of 5 years if the interest is compounded *continuously*.

Soln: $P = 23,200$, $r = 7\% = 0.07$, and $t = 5$. The expected amount in the account after 5 years is

$$A = 23,200 \cdot e^{0.07 \cdot 5} = \$32,922.37$$

note: we use a calculator to find the answer here!

Theorem (Exponential Equality)

For $a > 0$ and $a \neq 1$,

$$\text{if } a^{x_1} = a^{x_2}, \text{ then } x_1 = x_2$$

also

$$\text{if } x_1 = x_2, \text{ then } a^{x_1} = a^{x_2}$$

Example: Solve $\left(\frac{2}{3}\right)^x = \frac{9}{4}$ for x.

Soln: Try to use the above theorem.

$$\left(\frac{2}{3}\right)^x = \frac{9}{4} = \left(\frac{3}{2}\right)^2$$

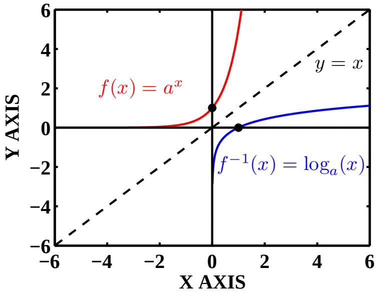
or

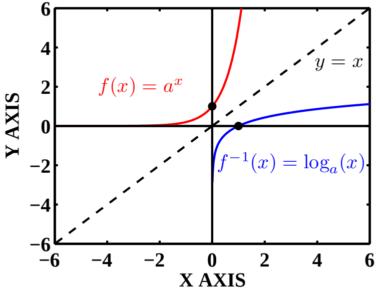
$$\left(\frac{2}{3}\right)^x = \left(\frac{2}{3}\right)^{-2} \text{ since } \left(\frac{3}{2}\right)^2 = \left(\frac{2}{3}\right)^{-2}$$

Then by the theorem on exponential equality, we must have that $x = -2$.

Consider the exponential function $f(x) = 2^x$. Like all exponential functions, f is one to one. Can a formula for $f^{-1}(x)$ be found? Using what we learned in section 2.5 on inverse functions:

1. Replace $f(x)$ with y: $y = 2^x$
2. Interchange x and y: $x = 2^y$
3. Solve for y: $y =$ the exponent to which we raise 2 get x.
4. Replace y with $f^{-1}(x)$: $f^{-1}(x) =$ the exponent to which we raise 2 get x.





We define a new symbol to replace the words "the exponent to which we raise 2 get x"

> $\log_2(x)$, ***read "the logarithm, base 2 of x" or "log, base 2, of x," means "the exponent to which we raise 2 get x."***

So, if $f(x) = 2^x$, then $f^{-1}(x) = \log_2(x)$. Note that $f^{-1}(8) = \log_2(8) = 3$, because 3 is the exponent to which we raise 2 get 8.

Examples:

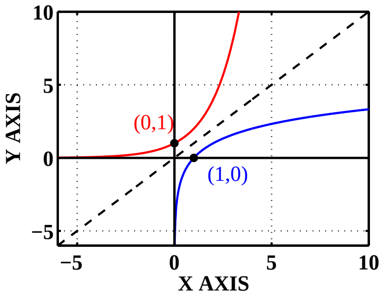
- $\log_2(32) = 5$, because 5 is the exponent to which we raise 2 get 32.
- $\log_2(1) = 0$, because 0 is the exponent to which we raise 2 get 1.

Definition (Logarithmic Function)

For $a > 0$ and $a \neq 1$, the logarithmic function with base a is denoted as $f(x) = \log_a(x)$, where

$$y = \log_a(x) \iff a^y = x$$

- Note that $\log_a(1) = 0$ for any base a, because $a^0 = 1$ for any base a.
- There are two bases that are used more frequently than others; they are 10 and e.
- The notation $\log_{10}(x)$ is abbreviated $\log(x)$ and $\log_e(x)$ is abbreviated $\ln(x)$. These are called the common logarithmic function and the natural logarithmic function, respectively.



The functions of $y = a^x$ and $y = \log_a(x)$ for $a > 0$ and $a \neq 1$ are inverse functions.

- So, the graph of $y = \log_a(x)$ is a reflection about the line $y = x$ of the graph of $y = a^x$.
- The graph of $y = a^x$ has the x-axis as its horizontal asymptote,
- while the graph of $y = \log_a(x)$ has the y-axis as its vertical asymptote.

Example: Graph $y = \log_5(x)$

Solution: If $y = \log_5(x)$ then $5^y = x$. We can find ordered pairs that are solutions by choosing y values and computing the x values.

x, or 5^y	y
	0
	1
	2
	-1
	-2

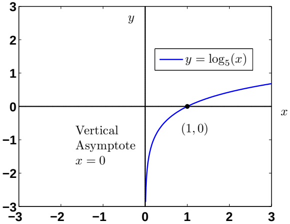
Solution: If $y = \log_5(x)$ then $5^y = x$. We can find ordered pairs that are solutions by choosing y values and computing the x values.

For $y = 0, x = 5^0 = 1$.
For $y = 1, x = 5^1 = 5$.
For $y = 2, x = 5^2 = 25$.
For $y = -1, x = 5^{-1} = \frac{1}{5}$.
For $y = -2, x = 5^{-2} = \frac{1}{25}$.

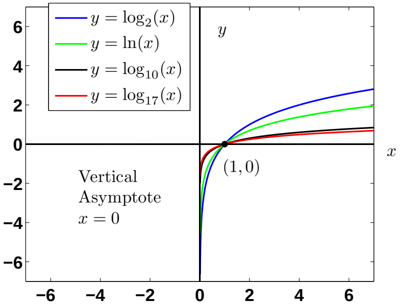
x, or 5^y	y
1	0
5	1
25	2
$\frac{1}{5}$	-1
$\frac{1}{25}$	-2

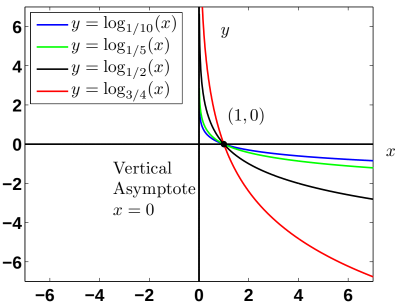
The preceding table of values shows the following:

$$\log_5(1) = 0$$
$$\log_5(5) = 1$$
$$\log_5(25) = 2$$
$$\log_5(\tfrac{1}{5}) = -1$$
$$\log_5(\tfrac{1}{25}) = -2$$



Properties of the Logarithmic Function $f(x) = \log_a(x)$





- $f(x)$ is an increasing function for $a > 1$ and a decreasing function whenever $0 < a < 1$.
- The x-intercept of the graph of f is (1, 0).
- The graph has the y-axis as a vertical asymptote.
- The domain of f is $(0, \infty)$, and the range of f is $(-\infty, \infty)$.
- The functions $f(x) = \log_a(x)$ and $f(x) = a^x$ are inverse functions.

Definition

Consider the function g defined by

$$g(x) = a \cdot f(x - c) + d \qquad \text{where } a, c, \text{ and } d \text{ are real numbers.}$$

Then

1. $g(x)$ is the "generalized" child graph of parent graph $f(x)$.
2. c represents the horizontal translation of f.
3. a reflection/magnification
4. d represents the vertical translation of f.

Definition (Multiple Transformations Graphing Algorithm)

Consider the function g defined by

$$g(x) = a \cdot f(x - c) + d \quad \text{where } a, c, \text{ and } d \text{ are real numbers.}$$

In order to graph $g(x)$ it is recommended to take the following steps:

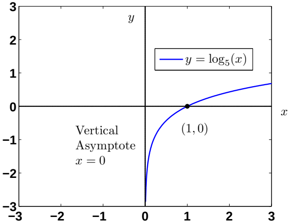
1. Identify and graph the parent graph $f(x)$, of $g(x)$.
2. (c) Translate (shift) f horizontally, i.e. apply $f(x \pm c)$.
3. (a) Apply the Reflection/magnification.
4. (d) Translate (shift) f vertically.

Note: If you are asked to graph, for example, $f(x) = -2\sqrt[3]{x+1} - 2$, then you should rename $f(x)$ and give it the new name of $g(x)$. Then find g's parent graph $f(x)$.

Use translations to graph: $g(x) = -2\log_5(x+1) - 1$

Solution: First identify and graph the parent graph $f(x) = \log_5(x)$, then apply the theory on graphical translations.

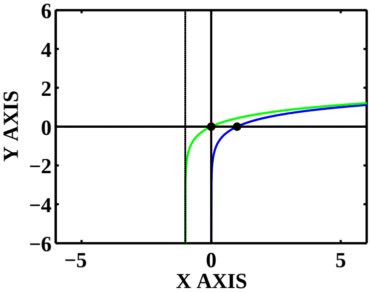
x, or 5^y	y
1	0
5	1
25	2
$\frac{1}{5}$	-1
$\frac{1}{25}$	-2



Step 2: Apply the horizontal shift. Graph:
$g_1(x) = \log_5(x+1) = f(x+1)$. Shift every point of the parent graph one unit left (horizontally) (NOTE: the vertical asymptote (VA) shifts one unit left to $x = -1$)

Step 2: Apply the horizontal shift. Graph: $g_1(x) = \log_5(x+1) = f(x+1)$. Shift every point of the parent graph one unit left (horizontally) (NOTE: the vertical asymptote (VA) shifts one unit left to $x = -1$)

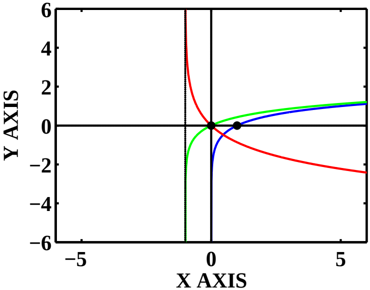
x, or 5^y	g_1
0	0
4	1
24	2
$-\frac{4}{5}$	-1
$-\frac{24}{25}$	-2



Step 3: Apply the reflection/magnification. Graph: $g_2(x) = -2\log_5(x+1) = -2 \cdot f(x+1)$. Now shift the points on the graph in step 2 by multiplying the y coordinate of each point by -2.

Step 3: Apply the reflection/magnification. Graph:
$g_2(x) = -2\log_5(x+1) = -2 \cdot f(x+1)$. Now shift the points on the graph in step 2 by multiplying the y coordinate of each point by -2.

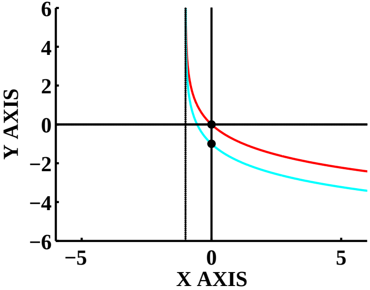
x, or 5^y	g_2
0	0
4	-2
24	-4
$-\frac{4}{5}$	2
$-\frac{24}{25}$	4



Step 4: Apply the vertical shift. Graph:
$g(x) = -2\log_5(x+1) - 1 = -2 \cdot f(x+1) - 1$.Translate each point on the graph in step 3 vertically downwards.

Step 4: Apply the vertical shift. Graph:
$g(x) = -2\log_5(x+1) - 1 = -2 \cdot f(x+1) - 1$. Translate each point on the graph in step 3 vertically downwards.

x, or 5^y	g_3
0	-1
4	-3
24	-5
$-\frac{4}{5}$	1
$-\frac{24}{25}$	3



$g(x) = -2\log_5(x+1) - 1 = -2 \cdot f(x+1) - 1.$

Theorem (One-to-One Property of Logarithms)

For $a > 0$ *and* $a \neq 1$,

if $\log_a(x_1) = \log_a(x_2)$*, then* $x_1 = x_2$.

Summary

Definition

Since exponential functions are one-to-one functions, they are invertible. The inverses of the exponential functions are called **logarithmic functions**.

- If $f(x) = a^x$, then $f^{-1}(x) = \log_a(x)$ for the inverse of the base-a exponential function.
- We read $\log_a(x)$ as "log of x base a," and we call the expression $\log_a(x)$ a logarithm.
- In general, $\log_a(x)$ is the exponent that is used on the base a to obtain the value x.
- Since the exponential function $f(x) = a^x$ has domain $(-\infty, \infty)$ and range $(0, \infty)$, the logarithmic function $f(x) = \log_a(x)$ has domain $(0, \infty)$ and range $(-\infty, \infty)$.
- So, there are no logarithms of negative numbers or zero.